

# Do hypergraphs have properties on chromatic polynomials not owned by graphs

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## Abstract

In this paper, we investigate chromatic polynomials of hypergraphs and find some properties which don't hold for chromatic polynomials of graphs. We first show that if  $z$  is a zero of the independence polynomial of a graph  $G$ , then  $1+1/z$  is a zero of the chromatic polynomial  $P(\mathcal{H}_{\bullet G}, \lambda)$ , where  $\mathcal{H}_{\bullet G}$  is the hypergraph obtained from  $G$  by adding a new vertex  $w$  and changing each edge  $\{u, v\}$  in  $G$  to an edge  $\{u, v, w\}$  in  $\mathcal{H}_{\bullet G}$ . Thus we deduce that chromatic polynomials of hypergraphs have all integers as their zeros and contain dense real zeros in the set of real numbers and dense complex zeros in the whole complex plane. We then prove that for any multigraph  $G = (V, E)$ , the number of totally cyclic orientations of  $G$  is equal to  $|P(\mathcal{H}_G, -1)|$ , where  $\mathcal{H}_G$  is the hypergraph  $(\mathcal{V}, \mathcal{E})$  with  $\mathcal{V} = V \cup E$  and  $\mathcal{E} = \{\{u_e, v_e, e\} : e \in E\}$ , where  $u_e$  and  $v_e$  are the two ends of  $e$ . Finally we show that the multiplicity of root "0" of  $P(\mathcal{H}, \lambda)$  may be at least 2 for some connected hypergraphs  $\mathcal{H}$ , and the multiplicity of root "1" of  $P(\mathcal{H}, \lambda)$  may be 1 for some connected and separable hypergraphs  $\mathcal{H}$ .

## 1 Introduction

A *hypergraph*  $\mathcal{H}$  consists of an order pair of vertex set  $\mathcal{V}$  and edge set  $\mathcal{E}$ , where  $\mathcal{E}$  is a subset of  $\{e \subseteq \mathcal{V} : |e| \geq 1\}$ . If  $|e| \leq 2$  for all  $e \in \mathcal{E}$ , then  $\mathcal{H}$  is a graph. For any integer  $\lambda \geq 1$ , a *proper  $\lambda$ -colouring* of a hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  is a mapping  $\phi : \mathcal{V} \rightarrow \{1, 2, \dots, \lambda\}$  such that  $|\{\phi(v) : v \in e\}| > 1$  holds for each  $e \in \mathcal{E}$ . Thus  $\mathcal{H}$  does not have any  $\lambda$ -colouring if  $|e| = 1$  for some edge  $e \in \mathcal{E}$ . Two proper  $\lambda$ -colourings  $\phi$  and  $\psi$  of  $\mathcal{H}$  are regarded as *distinct* if  $\phi(u) \neq \psi(u)$  for some vertex  $u$  in  $\mathcal{H}$ . Let

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$P(\mathcal{H}, \lambda)$  be the number of distinct proper  $\lambda$ -colourings of  $\mathcal{H}$ . This function  $P(\mathcal{H}, \lambda)$  is called *the chromatic polynomial of  $\mathcal{H}$* , and it is indeed a polynomial in  $\lambda$  of degree  $|\mathcal{V}|$ .

The graph-function  $P(\mathcal{H}, \lambda)$  for a hypergraph  $\mathcal{H}$  appeared in the work of Helgason [12] in 1972, and it is unknown if it had been introduced earlier. It has been studied extensively in the past twenty years by many researchers, such as Allagan [1, 2, 3], Borowiecki and Łazuka [5], Dohmen [7], Tomescu [18, 19, 20, 21] and Walter [23]. They extended many properties of chromatic polynomials of graphs on computations, expressions, factorizations, etc, to chromatic polynomials of hypergraphs.

In this article, we will present some properties on chromatic polynomials of hypergraphs which are different from the following important properties on chromatic polynomials of graphs:

- (i)  $(-\infty, 0)$ ,  $(0, 1)$  and  $(1, 32/27]$  are zero-free intervals for chromatic polynomials of graphs;
- (ii)  $|P(G, -1)|$  counts the number of acyclic orientations (i.e., orientations without any directed cycle) of  $G$ ;
- (iii)  $P(G, \lambda)$  has no factor  $\lambda^2$  whenever  $G$  is connected;
- (iv)  $P(G, \lambda)$  has a factor  $(\lambda - 1)^2$  whenever  $G$  is connected and separable.

For any simple graph  $G = (V, E)$  (i.e., it has no loops nor parallel edges), let  $\mathcal{H}_{\bullet G}$  be the hypergraph with vertex set  $\mathcal{V} = V \cup \{w\}$  and edge set  $\mathcal{E} = \{\{u, v, w\} : uv \in E\}$ , i.e.,  $\mathcal{H}_{\bullet G}$  is obtained from  $G$  by adding a new vertex  $w$  and changing each edge  $\{u, v\}$  in  $G$  to an edge  $\{u, v, w\}$  in  $\mathcal{H}_{\bullet G}$ .

*The independence polynomial* of a graph  $G$  is defined to be  $I(G, x) = \sum_A x^{|A|}$ , where the sum runs over all independent sets  $A$  of  $G$ . One of the main purposes in this article is to establish a relation between  $P(\mathcal{H}_{\bullet G}, \lambda)$  and  $I(G, x)$  in the following theorem.

**Theorem 1.** *For any simple graph  $G$  of order  $n$ ,*

$$P(\mathcal{H}_{\bullet G}, \lambda) = \lambda(\lambda - 1)^n I(G, 1/(\lambda - 1)). \quad (1)$$

Theorem 1 implies that the multiplicity of root “1” in  $P(\mathcal{H}_{\bullet G}, \lambda)$  is  $n - \alpha(G)$ , where  $\alpha(G)$  is the independence number of  $G$ , and whenever  $z$  is a zero of  $I(G, x)$ ,  $1 + 1/z$  is a zero of  $P(\mathcal{H}_{\bullet G}, \lambda)$ . If  $G$  is the complete graph  $K_n$ , then  $I(G, x) = 1 + nx$  and by Theorem 1,  $1 - n$  is a zero of  $P(\mathcal{H}_{\bullet G}, \lambda)$ .

Brown, Hickman and Nowakowski [4] showed that real roots of independence polynomials are dense in  $(-\infty, 0]$  while the complex roots of these polynomials are dense in  $\mathbb{C}$  (i.e. the whole complex plane). Chudnovsky and Seymour [6] proved that if  $G$  is

clawfree, then all the roots of its independence polynomial are real. By Theorem 1 and results in [4, 6], we have the following conclusions immediately except Corollary 1 (b) whose proof will be given in Section 3.

**Corollary 1.** (a) *The complex roots of  $P(\mathcal{H}_{\bullet G}, \lambda)$  for all graphs  $G$  are dense in the whole complex plane;*

(b) *The real roots of  $P(\mathcal{H}_{\bullet G}, \lambda)$  for all graphs  $G$  are dense in the set of real numbers;*

(c) *Every negative integer is a root of  $P(\mathcal{H}_{\bullet G}, \lambda)$  for some graph  $G$ ;*

(d) *If  $G$  is clawfree, then all roots of  $P(\mathcal{H}_{\bullet G}, \lambda)$  are real.*

For a given multigraph  $G = (V, E)$ ,  $\mathcal{H}_G = (\mathcal{V}, \mathcal{E})$  is another hypergraph constructed from  $G$ , where  $\mathcal{V} = V \cup \{w_e : e \in E\}$  and  $\mathcal{E} = \{\{u_e, v_e, w_e\} : e \in E\}$ , where  $u_e$  and  $v_e$  are the two ends of  $e$ , i.e.,  $\mathcal{H}_G$  is obtained from  $G$  by adding  $|E|$  new vertices  $\{w_e : e \in E\}$  and changing each edge  $e$  in  $G$  to an edge  $\{u_e, v_e, w_e\}$  in  $\mathcal{H}_G$ . We will express  $P(\mathcal{H}_G, \lambda)$  in terms of the Tutte polynomial  $T_G(x, y)$  for  $G = (V, E)$ , where  $T_G(x, y)$  is defined below:

$$T_G(x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)}, \quad (2)$$

where  $r(A) = |V| - c(A)$  and  $c(A)$  is the number of components of the spanning subgraph  $(V, A)$  of  $G$  for any  $A \subseteq E$ .

**Theorem 2.** *For any multigraph  $G = (V, E)$  with order  $n$  and size  $m$ ,*

$$P(\mathcal{H}_G, \lambda) = \lambda^{m - n + 2c(G)} \cdot (-1)^{n + c(G)} \cdot T_G(1 - \lambda^2, (\lambda - 1)/\lambda), \quad (3)$$

where  $c(G)$  is the number of components of  $G$ , i.e.,  $c(G) = c(E)$ .

Stanley [15] showed that for any multigraph  $G$ ,  $|P(G, -1)| = T_G(2, 0)$  counts the number of acyclic orientations of  $G$ , where an acyclic orientation of  $G$  is an orientation of all edges in  $G$  such that the digraph obtained does not have any directed cycle. By Theorem 2, the number of totally cyclic orientations of  $G$  (i.e., orientations on which each arc is in some cycle) can be determine by the value of  $|P(\mathcal{H}_G, -1)|$ .

**Corollary 2.** *For any multigraph  $G$ ,  $|P(\mathcal{H}_G, -1)| = T_G(0, 2)$  counts the number of totally cyclic orientations of  $G$ .*

It is well known that for any connected graph  $G$ ,  $P(G, \lambda)$  has a factor  $\lambda$  but no factor  $\lambda^2$ . However,  $P(\mathcal{H}, \lambda)$  may have a factor  $\lambda^2$  for a connected hypergraph  $\mathcal{H}$ .

A hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  is said to be *connected* if for any two vertices  $v_1, v_2$  in  $\mathcal{H}$ , there exists a sequence of edges  $e_0, e_1, \dots, e_k$  in  $\mathcal{H}$  such that  $v_1 \in e_0$ ,  $v_2 \in e_k$  and

$e_i \cap e_{i+1} \neq \emptyset$  holds for all  $i = 0, 1, \dots, k-1$ . Now assume that  $\mathcal{H}$  is connected. An edge  $e$  in  $\mathcal{H}$  is called as a *bridge* of  $\mathcal{H}$  if  $\mathcal{H} - e$  (i.e., the hypergraph obtained from  $\mathcal{H}$  by removing  $e$ ) is disconnected. Let  $B(\mathcal{H})$  be the set of bridges of  $\mathcal{H}$ . If  $\mathcal{H}$  is a graph (i.e.,  $|e| = 2$  for all  $e \in \mathcal{E}$ ), then the spanning subgraph  $(\mathcal{V}, B(\mathcal{H}))$  is connected if and only if  $B(\mathcal{H}) = \mathcal{E}$  and  $\mathcal{H}$  is a tree. However, if  $\mathcal{H}$  has some edge  $e$  with  $|e| \geq 3$ , it is possible that  $(\mathcal{V}, B(\mathcal{H}))$  is connected while  $B(\mathcal{H})$  is a proper subset of  $\mathcal{E}$ . Such an example is given in Figure 1 (a).

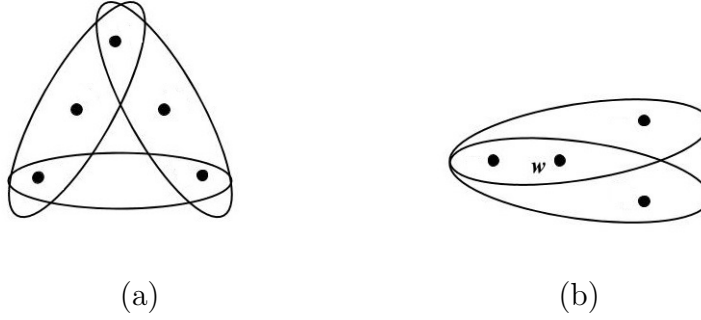


Figure 1: Two hypergraphs

**Theorem 3.** *Let  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  be any connected hypergraph. If  $B(\mathcal{H})$  is a proper subset of  $\mathcal{E}$  and the sub-hypergraph  $(\mathcal{V}, B(\mathcal{H}))$  is connected, then  $\lambda^2$  is a factor of  $P(\mathcal{H}, \lambda)$ .*

It is well known that for a connected graph  $G$ , if  $G$  is separable (i.e.,  $G$  has a cut-vertex), then  $(\lambda - 1)^2$  is a factor of  $P(G, \lambda)$ . Can this property be extended to hypergraphs?

We first need to make it clear what are separable hypergraphs. A connected hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  is said to be *separable* at a vertex  $w$  if the hypergraph  $\mathcal{H} - w$  obtained from  $\mathcal{H}$  by removing  $w$  and all edges containing  $w$  is disconnected. This definition is a natural extension of the one for separable graphs. Observe that  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  is separable at  $w$  if and only if  $\mathcal{V}$  has two subsets  $\mathcal{V}_1$  and  $\mathcal{V}_2$  such that  $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ ,  $\mathcal{V}_1 \cap \mathcal{V}_2 = \{w\}$  and for each  $e \in \mathcal{E}$ , either  $w \in e$  or  $e \subseteq \mathcal{V}_i$  for some  $i \in \{1, 2\}$ . Note that if  $\mathcal{H}$  is a graph, then  $w \in e$  implies that  $e \subseteq \mathcal{V}_i$  for some  $i$ . But if  $\mathcal{H}$  is not a graph, it is possible that  $|e \cap \mathcal{V}_i| \geq 2$  for both  $i \in \{1, 2\}$ .

The hypergraph in Figure 1 (b) is connected and separable at  $w$ , but its chromatic polynomial is  $\lambda(\lambda-1)(\lambda^2+\lambda-1)$  which does not contain a factor  $(\lambda-1)^2$ . Actually this hypergraph is contained in a family of connected and separable hypergraphs whose chromatic polynomials have no factor  $(\lambda-1)^2$ . For any  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ , let  $F(\mathcal{H})$  be the set of vertices  $w \in \mathcal{V}$  such that  $w \in e$  for every  $e \in \mathcal{E}$ . Observe that  $w \in F(\mathcal{H})$  if and only if  $e \not\subseteq \mathcal{V} - \{w\}$  for every  $e \in \mathcal{E}$ . If  $\mathcal{H}$  does not have parallel edges (i.e., edges  $e_1, e_2$  with  $e_1 = e_2$ ), then  $F(\mathcal{H}) = \mathcal{V}$  if and only if  $\mathcal{H}$  is connected and  $|\mathcal{E}| = 1$ . It is

trivial that if  $|\mathcal{E}| = 1$ , then  $P(\mathcal{H}, \lambda)$  does not have a factor  $(\lambda - 1)^2$ . We will show that if  $|\mathcal{E}| \geq 2$  and  $F(\mathcal{H}) \neq \emptyset$ , then  $P(\mathcal{H}, \lambda)$  has a factor  $(\lambda - 1)^2$  if and only if  $|F(\mathcal{H})| = 1$ .

For a hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  and any  $\mathcal{V}_0 \subseteq \mathcal{V}$ , let  $\mathcal{H} \cdot \mathcal{V}_0$  denote the hypergraph obtained from  $\mathcal{H}$  by identifying all vertices in  $\mathcal{V}_0$  as one vertex and let  $\mathcal{H}[\mathcal{V}_0]$  be the hypergraph with vertex set  $\mathcal{V}_0$  and edge set  $\{e \in \mathcal{E} : e \subseteq \mathcal{V}_0\}$ . We call  $\mathcal{H}[\mathcal{V}_0]$  the sub-hypergraph of  $\mathcal{H}$  induced by  $\mathcal{V}_0$ . Let  $\mathcal{H} - \mathcal{V}_0$  be the induced sub-hypergraph  $\mathcal{H}[\mathcal{V} - \mathcal{V}_0]$ . A hypergraph is said to be empty if it contains no edges. Let  $\mathcal{I}(\mathcal{H})$  be the set of those subsets  $\mathcal{V}_0$  of  $\mathcal{V}$  such that  $\mathcal{H}[\mathcal{V}_0]$  is an empty graph. A hypergraph  $\mathcal{H}$  is said to be *Sperner* if  $e_1 \not\subseteq e_2$  for each pair of edges  $e_1, e_2$  in  $\mathcal{H}$ .

For the case that  $F(\mathcal{H}) = \emptyset$  and  $\mathcal{H}$  is separable, we also give an equivalent statement for  $P(\mathcal{H}, \lambda)$  to have a factor  $(\lambda - 1)^2$ .

**Theorem 4.** *Let  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  be any connected and Sperner hypergraph with  $|\mathcal{E}| \geq 2$ .*

- (i) *If  $F(\mathcal{H}) \neq \emptyset$ , then  $P(\mathcal{H}, \lambda)$  has a factor  $(\lambda - 1)^2$  if and only if  $|F(\mathcal{H})| = 1$ ;*
- (ii) *If  $F(\mathcal{H}) = \emptyset$  and  $\mathcal{H}$  has a vertex  $w$  and two proper subsets  $\mathcal{V}_1$  and  $\mathcal{V}_2$  of  $\mathcal{V}$  such that  $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ ,  $\mathcal{V}_1 \cap \mathcal{V}_2 = \{w\}$  and for each  $e \in \mathcal{E}$ , either  $w \in e$  or  $e \subseteq \mathcal{V}_i$  for some  $i$ , then  $P(\mathcal{H}, \lambda)$  has a factor  $(\lambda - 1)^2$  if and only if one of the following conditions is satisfied:*
  - (a)  $\mathcal{V}_i \notin \mathcal{I}(\mathcal{H})$  for both  $i = 1, 2$ ;
  - (b) for some  $i \in \{1, 2\}$ ,  $\mathcal{V}_i \in \mathcal{I}(\mathcal{H})$ ,  $\mathcal{V}_{3-i} \notin \mathcal{I}(\mathcal{H})$  and  $P(\mathcal{H} \cdot \mathcal{V}_i, \lambda)$  has a factor  $(\lambda - 1)^2$ .

We will prove Theorems 1-4 in Sections 3-5 after some fundamental results are introduced in Section 2. We will also propose some open problems regarding multiplicities of roots “0” and “1” of  $P(\mathcal{H}, \lambda)$  for a hypergraph  $\mathcal{H}$ .

## 2 Preliminary

In this section, we present several known results on chromatic polynomials of hypergraphs, which will be applied later. The first one follows directly from the definition of proper colouring of a hypergraph.

**Proposition 1.** *Let  $e_1, e_2$  be any two edges in a hypergraph  $\mathcal{H}$ . If  $e_1 \subseteq e_2$ , then*

$$P(\mathcal{H}, \lambda) = P(\mathcal{H} - e_2, \lambda),$$

*where  $\mathcal{H} - e_2$  is the hypergraph obtained from  $\mathcal{H}$  by removing  $e_2$ .*

By Proposition 1, we need only to consider Sperner hypergraphs in the function  $P(\mathcal{H}, \lambda)$ .

For a hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ , a *component* of  $\mathcal{H}$  is an induced and connected sub-hypergraph  $\mathcal{H}[\mathcal{V}_0]$  such that  $\mathcal{H}[\mathcal{V}_0 \cup \{v\}]$  is disconnected for any  $v \in \mathcal{V} - \mathcal{V}_0$ . By the definition of  $P(\mathcal{H}, \lambda)$ , we have the following expression for  $P(\mathcal{H}, \lambda)$  when  $\mathcal{H}$  is disconnected.

**Proposition 2.** *Assume that  $\mathcal{H}_1, \dots, \mathcal{H}_k$  are components of  $\mathcal{H}$ . Then*

$$P(\mathcal{H}, \lambda) = \prod_{1 \leq i \leq k} P(\mathcal{H}_i, \lambda).$$

For any hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  and  $\mathcal{V}_0 \subset \mathcal{V}$ , recall that  $\mathcal{H} \cdot \mathcal{V}_0$  is obtained from  $\mathcal{H}$  by identifying all vertices in  $\mathcal{V}_0$  as one, i.e.,  $\mathcal{H} \cdot \mathcal{V}_0$  is the hypergraph with vertex set  $(\mathcal{V} - \mathcal{V}_0) \cup \{w\}$  and edge set

$$\{e' \in \mathcal{E} : e' \cap \mathcal{V}_0 = \emptyset\} \cup \{(e' - \mathcal{V}_0) \cup \{w\} : e' \cap \mathcal{V}_0 \neq \emptyset\},$$

where  $w \notin \mathcal{V}$ . For an edge  $e$  in  $\mathcal{H}$ , let  $\mathcal{H}/e$  be the hypergraph  $(\mathcal{H} - e) \cdot e$ . This hypergraph  $\mathcal{H}/e$  is said to be obtained from  $\mathcal{H}$  by *contracting*  $e$ .

The deletion-contraction formula for chromatic polynomials of graphs is very important for the study of this polynomial. It was extended to chromatic polynomials of hypergraphs by Jones [14].

**Theorem 5** ([14]). *Let  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  be a hypergraph. For any  $e \in \mathcal{E}$ ,*

$$P(\mathcal{H}, \lambda) = P(\mathcal{H} - e, \lambda) - P(\mathcal{H}/e, \lambda). \quad (4)$$

Note that Theorem 5 can be equivalently stated below: for any subset  $e$  of  $\mathcal{V}$ ,

$$P(\mathcal{H}, \lambda) = P(\mathcal{H} + e, \lambda) + P(\mathcal{H} \cdot e, \lambda), \quad (5)$$

where  $\mathcal{H} + e$  is the hypergraph obtained from  $H$  by adding a new edge  $e$ .

A hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  is written as  $\mathcal{H}_1 \cup \mathcal{H}_2$ , where  $\mathcal{H}_i = (\mathcal{V}_i, \mathcal{E}_i)$  is a hypergraph for  $i = 1, 2$ , if  $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ ,  $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$  and for any  $e \subseteq \mathcal{V}_1 \cap \mathcal{V}_2$ ,  $e \in \mathcal{E}_1$  if and only if  $e \in \mathcal{E}_2$ . If  $\{u, v\} \in \mathcal{E}_1 \cap \mathcal{E}_2$  for each pair  $\{u, v\} \subseteq \mathcal{V}_1 \cap \mathcal{V}_2$ , then write  $\mathcal{H}_1 \cap \mathcal{H}_2 = K_p$ , where  $p = |\mathcal{E}_1 \cap \mathcal{E}_2|$ . Borowiecki and Łazuka [5] extended Zykov's result [24] on the chromatic polynomial of the graph  $G_1 \cup G_2$  with  $G_1 \cap G_2 \cong K_p$ .

**Theorem 6** ([5]). *If  $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$  and  $\mathcal{H}_1 \cap \mathcal{H}_2 = K_p$ , then*

$$P(\mathcal{H}, \lambda) = \frac{P(\mathcal{H}_1, \lambda)P(\mathcal{H}_2, \lambda)}{P(K_p, \lambda)}. \quad (6)$$

### 3 Proof of Theorem 1 and Corollary 1 (b)

Let  $G = (V, E)$  be a simple graph, i.e.,  $G$  has neither parallel edges nor loops. For  $v \in V$ , let  $N_G(v)$  (or simply  $N(v)$ ) be the set  $\{u \in V : uv \in E\}$ , and let  $N[v] = N(v) \cup \{v\}$ . The degree of  $v$  in  $G$ , denoted by  $d(v)$ , is the size  $|N(v)|$  of  $N(v)$ .

**Proposition 3.** *For any vertex  $v \in V$ ,*

$$P(\mathcal{H}_{\bullet G}, \lambda) = (\lambda - 1) \cdot P(\mathcal{H}_{\bullet G - \{v\}}, \lambda) + (\lambda - 1)^{d(v)} \cdot P(\mathcal{H}_{\bullet G - N[v]}, \lambda). \quad (7)$$

*Proof.* Let  $w$  be the new vertex in  $\mathcal{H}_{\bullet G}$  when it is produced from  $G$ . By (5),

$$P(\mathcal{H}_{\bullet G}, \lambda) = P(\mathcal{H}_{\bullet G} + e, \lambda) + P(\mathcal{H}_{\bullet G} \cdot e, \lambda), \quad (8)$$

where  $e = \{w, v\}$ .

Observe that  $e \subset \{w, v, v_i\}$  for all  $v_i \in N(v)$ . By Proposition 1,

$$P(\mathcal{H}_{\bullet G} + e, \lambda) = P(\mathcal{H}_{\bullet G} + e - \mathcal{E}(v), \lambda),$$

where  $\mathcal{E}(v) = \{\{w, v, v_i\} : v_i \in N(v)\}$ . By Theorem 6,

$$P(\mathcal{H}_{\bullet G} + e - \mathcal{E}(v), \lambda) = (\lambda - 1) \cdot P(\mathcal{H}_{\bullet G} - \{v\}, \lambda) = (\lambda - 1) \cdot P(\mathcal{H}_{\bullet G - \{v\}}, \lambda). \quad (9)$$

Note that the edges  $\{w, v, v_i\}$  in  $\mathcal{H}_{\bullet G}$ , where  $v_i \in N(v)$ , are changed to  $\{w, v_i\}$  in  $\mathcal{H}_{\bullet G} \cdot e$ , and thus all edges  $\{w, v_i, u\}$  in  $\mathcal{H}_{\bullet G}$ , where  $u \in N(v_i) - \{v\}$ , can be removed by Proposition 1. By Theorem 6 again,

$$P(\mathcal{H}_{\bullet G} \cdot e, \lambda) = (\lambda - 1)^{d(v)} \cdot P(\mathcal{H}_{\bullet G - N[v]}, \lambda). \quad (10)$$

Hence the result follows from (8), (9) and (10).  $\square$

The following property on the independence polynomial of a graph is needed for proving Theorem 1.

**Proposition 4** ([6]). *Let  $G = (V, E)$  be any simple graph and  $v \in V$ . Then  $I(G, x) = I(G - \{v\}, x) + xI(G - N[v], x)$ .*

Now we are ready to prove Theorem 1 by applying Propositions 3 and 4.

*Proof of Theorem 1:* Suppose that the result fails. Assume that  $G = (V, E)$  is a simple graph for which the result fails and  $|V| + |E|$  has the minimum value among all those graphs for which the result fails. We shall complete the proof by showing the following claims.

**Claim 1:**  $n = |V| \geq 2$ .

Assume that  $n = |V| = 1$ . Then  $E = \emptyset$  as  $G$  is simple. Thus  $\mathcal{H}_{\bullet G}$  is the hypergraph with two vertices and no edges, implying that  $P(\mathcal{H}_{\bullet G}, \lambda) = \lambda^2$ . Observe that the right-hand side of (1) is

$$\lambda(\lambda - 1)(1 + 1/(\lambda - 1)) = \lambda^2,$$

implying that Theorem 1 holds for this graph, a contradiction.

**Claim 2:**  $G$  does not exist.

Let  $v$  be any vertex of  $G$ , by Proposition 3, we have

$$P(\mathcal{H}_{\bullet G}, \lambda) = (\lambda - 1) \cdot P(\mathcal{H}_{\bullet(G-\{v\})}, \lambda) + (\lambda - 1)^{d(v)} \cdot P(\mathcal{H}_{\bullet(G-N[v])}, \lambda). \quad (11)$$

By the assumption on  $G$ , Theorem 1 holds for both  $G - \{v\}$  and  $G - N[v]$ . Thus

$$P(\mathcal{H}_{\bullet(G-\{v\})}, \lambda) = \lambda \cdot (\lambda - 1)^{n-1} \cdot I(G - v, \frac{1}{\lambda - 1}) \quad (12)$$

and

$$P(\mathcal{H}_{\bullet(G-N[v])}, \lambda) = \lambda \cdot (\lambda - 1)^{n-d(v)-1} \cdot I(G - N[v], \frac{1}{\lambda - 1}). \quad (13)$$

From Proposition 4 and equalities (11), (12) and (13), we obtain

$$\begin{aligned} P(\mathcal{H}_{\bullet G}, \lambda) &= \lambda(\lambda - 1)^n \cdot I(G - \{v\}, \frac{1}{\lambda - 1}) \\ &\quad + \lambda \cdot (\lambda - 1)^{n-1} \cdot I(G - N[v], \frac{1}{\lambda - 1}) \\ &= \lambda \cdot (\lambda - 1)^n \cdot I(G, \frac{1}{\lambda - 1}). \end{aligned}$$

Thus equality (1) holds for  $G$ , a contradiction. Therefore Claim 2 is proved and Theorem 1 holds.  $\square$

We end this section by providing a proof of Corollary 1 (b).

*Proof of Corollary 1 (b).* It has been shown in [4] that the real roots of independence polynomials are dense in the interval  $(-\infty, 0]$ . Then Theorem 1 implies that the real roots of chromatic polynomials of hypergraphs  $\mathcal{H}_{\bullet G}$  for all graphs  $G$  are dense in the interval  $(-\infty, 1]$ . By Corollary 4, which follows from Proposition 8 in Section 5 directly, the real roots of the chromatic polynomials of hypergraphs  $\mathcal{H}_{\bullet G} + K_1$  for all graphs  $G$  are dense in the interval  $(-\infty, 2]$ , where  $\mathcal{H}_{\bullet G} + K_1$  is the hypergraph obtained from  $\mathcal{H}_{\bullet G}$  by adding a new vertex  $u$  and adding new edges  $\{u, v\}$  for all vertices  $v$  in  $\mathcal{H}_{\bullet G}$ . Repeating this process or applying the fact that the real roots of chromatic polynomials of graphs are dense in  $[2, \infty)$ , the result of Corollary 1 (b) holds.  $\square$



## 4 Proof of Theorem 2

Let  $G = (V, E)$  be a graph of order  $n$  and size  $m$ . Assume that  $G$  may have loops or parallel edges. We first establish the following recursive formula for  $P(\mathcal{H}_G, \lambda)$ .

**Proposition 5.** *For any  $e \in E(G)$ ,*

$$P(\mathcal{H}_G, \lambda) = \lambda P(\mathcal{H}_{G-e}, \lambda) - P(\mathcal{H}_{G/e}, \lambda). \quad (14)$$

*Proof.* Assume that  $u$  and  $v$  are the two ends of  $e$  in  $G$ . It is possible that  $u = v$ , as  $e$  may be a loop. Let  $e' = \{u, v, e\}$ . So  $e'$  is the edge in  $\mathcal{H}_G$  corresponding to  $e$ . By Theorem 5, we have

$$P(\mathcal{H}_G, \lambda) = P(\mathcal{H}_G - e', \lambda) - P(\mathcal{H}_G/e', \lambda) \quad (15)$$

Note that  $\mathcal{H}_G - e'$  consists of an isolated vertex and the hypergraph  $\mathcal{H}_{G-e}$ , and  $\mathcal{H}_G/e'$  is actually the hypergraph  $\mathcal{H}_{G/e}$ . Thus the result holds.  $\square$

Listed below are some other properties of  $P(\mathcal{H}_G, \lambda)$  which can be proved easily by applying Theorem 6 and Proposition 5.

**Proposition 6.** *Let  $G$  be a multigraph of order  $n$ .*

- (a) *If  $G$  is an empty graph, then  $P(\mathcal{H}_G, \lambda) = \lambda^n$ ;*
- (b) *If  $e$  is a loop of  $G$ , then  $P(\mathcal{H}_G, \lambda) = (\lambda - 1)P(\mathcal{H}_{G-e}, \lambda)$ ;*
- (c) *If  $e$  is a bridge of  $G$ , then  $P(\mathcal{H}_G, \lambda) = (\lambda^2 - 1)P(\mathcal{H}_{G/e}, \lambda)$ .*

Some fundamental properties on Tutte polynomials  $T_G(x, y)$  are needed for proving Theorem 2.

**Proposition 7** ([10, 22]). *Let  $G$  be a multigraph.*

- (a) *If  $G$  is an empty graph, then  $T_G(x, y) = 1$ ;*
- (b) *If  $e$  is a loop of  $G$ , then  $T_G(x, y) = y \cdot T_{G-e}(x, y)$ ;*
- (c) *If  $e$  is a bridge of  $G$ , then  $T_G(x, y) = x \cdot T_{G/e}(x, y)$ ;*
- (d) *For any  $e \in E(G)$ , if  $e$  is neither a loop nor a bridge, then  $T_G(x, y) = T_{G-e}(x, y) + T_{G/e}(x, y)$ .*

We are now ready to prove Theorem 2.

*Proof of Theorem 2:* We will prove the result by induction on the size  $m$  of  $G$ .

If  $m = 0$ , Theorem 2 holds for  $G$  by Propositions 6 (a) and 7 (a).

Assume that Theorem 2 holds for any graph of size less than  $m$ , where  $m > 0$ . Now we assume that  $G = (V, E)$  is a graph of size  $m$ . Let  $e$  be any edge in  $G$ .

**Case 1:**  $e$  is a loop.

By Propositions 6 (b) and 7 (b),

$$P(\mathcal{H}_G, \lambda) = (\lambda - 1)P(\mathcal{H}_{G-e}, \lambda), \quad T_G(x, y) = yT_{G-e}(x, y). \quad (16)$$

By the inductive assumption, Theorem 2 holds for  $G - e$ , i.e.,

$$P(\mathcal{H}_{G-e}, \lambda) = \lambda^{m-1-n+2c(G)} \cdot (-1)^{n+c(G)} \cdot T_{G-e}(1 - \lambda^2, (\lambda - 1)/\lambda). \quad (17)$$

Thus Theorem 2 holds for  $G$  by equalities in (16) and (17).

**Case 2:**  $e$  is a bridge.

By Propositions 6 (c) and 7 (c),

$$P(\mathcal{H}_G, \lambda) = (\lambda^2 - 1)P(\mathcal{H}_{G/e}, \lambda), \quad T_G(x, y) = xT_{G/e}(x, y). \quad (18)$$

By the inductive assumption, Theorem 2 holds for  $G/e$ , i.e.,

$$P(\mathcal{H}_{G/e}, \lambda) = \lambda^{m-n+2c(G)} \cdot (-1)^{n-1+c(G)} \cdot T_{G/e}(1 - \lambda^2, (\lambda - 1)/\lambda). \quad (19)$$

Thus Theorem 2 holds for  $G$  by equalities in (18) and (19).

**Case 3:**  $e$  is neither a bridge nor a loop.

Then  $c(G - e) = c(G/e) = c(G)$ . By inductive assumption, Theorem 2 holds for  $G - e$  and  $G/e$ , i.e.,

$$P(\mathcal{H}_{G-e}, \lambda) = \lambda^{m-1-n+2c(G)} \cdot (-1)^{n+c(G)} \cdot T_{G-e}(1 - \lambda^2, (\lambda - 1)/\lambda) \quad (20)$$

and

$$P(\mathcal{H}_{G/e}, \lambda) = \lambda^{m-n+2c(G)} \cdot (-1)^{n-1+c(G)} \cdot T_{G/e}(1 - \lambda^2, (\lambda - 1)/\lambda). \quad (21)$$

By (20), (21) and Proposition 7 (d), it can be verified that Theorem 2 holds for  $G$ .  $\square$

## 5 Proofs of Theorems 3 and 4

In this section, we will complete the proofs of Theorems 3 and 4.

*Proof of Theorem 3:* By a result due to Tomescu [18], the coefficient of  $\lambda$  in  $P(\mathcal{H}, \lambda)$  is equal to

$$a_1 = \sum_j (-1)^j N_j$$

where  $N_j$  is the number of connected and spanning sub-hypergraphs of  $\mathcal{H}$  with exactly  $j$  edges. By the given conditions, any sub-hypergraph  $(\mathcal{V}, \mathcal{E}')$  of  $\mathcal{H}$  is connected if and only if  $B(\mathcal{H}) \subseteq \mathcal{E}'$ . Assume that  $r = |B(\mathcal{H})|$  and  $k = |\mathcal{E}| - |B(\mathcal{H})| > 0$ . Then  $N_j = \binom{k}{j-r}$  and

$$a_1 = \sum_{j=r}^{r+k} (-1)^j \binom{k}{j-r} = 0.$$

Thus the result holds.  $\square$

Some results are needed for proving Theorem 4. Let  $\Phi(\mathcal{H})$  be the set of those partitions  $\{\mathcal{V}_1, \dots, \mathcal{V}_k\}$  of  $\mathcal{V}$  such that each  $\mathcal{V}_i$  is a non-empty member in  $\mathcal{I}(\mathcal{H})$ . By the definition of  $P(\mathcal{H}, \lambda)$ ,

$$P(\mathcal{H}, \lambda) = \sum_{\{\mathcal{V}_1, \dots, \mathcal{V}_k\} \in \Phi(\mathcal{H})} (\lambda)_k, \quad (22)$$

where  $(x)_0 = 1$  and  $(x)_k = x(x-1)\cdots(x-k+1)$  for any number  $x$  and positive integer  $k$ . By (22), we can deduce the following result.

**Proposition 8.** *Let  $w$  be a fixed vertex in  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ . Then*

$$P(\mathcal{H}, \lambda) = \lambda \sum_{w \in \mathcal{V}_0 \in \mathcal{I}(\mathcal{H})} P(\mathcal{H} - \mathcal{V}_0, \lambda - 1). \quad (23)$$

*Proof.* By (22), we can assume that

$$P(\mathcal{H}, \lambda) = \sum_{\substack{\{\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_k\} \in \Phi(\mathcal{H}) \\ w \in \mathcal{V}_0}} (\lambda)_{k+1}. \quad (24)$$

For any  $\mathcal{V}_0 \in \mathcal{I}(\mathcal{H})$  with  $w \in \mathcal{V}_0$ ,  $\{\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_k\} \in \Phi(\mathcal{H})$  if and only if  $\{\mathcal{V}_1, \dots, \mathcal{V}_k\} \in \Phi(\mathcal{H} - \mathcal{V}_0)$ . Thus

$$P(\mathcal{H}, \lambda) = \lambda \sum_{w \in \mathcal{V}_0 \in \mathcal{I}(\mathcal{H})} \sum_{\{\mathcal{V}_1, \dots, \mathcal{V}_k\} \in \Phi(\mathcal{H} - \mathcal{V}_0)} (\lambda - 1)_k. \quad (25)$$

By (22) again, the result holds.  $\square$

By Proposition 8,  $\lambda$  is a factor of  $P(\mathcal{H}, \lambda)$  for any hypergraph  $\mathcal{H}$ . Actually  $\lambda - 1$  is also a factor of  $P(\mathcal{H}, \lambda)$  whenever  $\mathcal{H}$  is not an empty graph.

**Corollary 3.** *For any hypergraph  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ ,  $\lambda^{c(\mathcal{H})}$  and  $(\lambda - 1)^{c'(\mathcal{H})}$  are factors of  $P(\mathcal{H}, \lambda)$ , where  $c(\mathcal{H})$  is the number of components of  $\mathcal{H}$  and  $c'(\mathcal{H})$  is the number of those components of  $\mathcal{H}$  which contain edges.*

*Proof.* We just prove that  $(\lambda - 1)^{c'(\mathcal{H})}$  is a factor of  $P(\mathcal{H}, \lambda)$ . It suffices to show that if  $\mathcal{H}$  is connected and non-empty, then  $\lambda - 1$  is a factor of  $P(\mathcal{H}, \lambda)$ . As  $\mathcal{H}$  is not empty,  $\mathcal{V} \notin \mathcal{I}(\mathcal{H})$ . Thus  $P(\mathcal{H} - \mathcal{V}_0, \lambda)$  has a factor  $\lambda$  for every  $\mathcal{V}_0 \in \mathcal{I}(\mathcal{H})$ . By Proposition 8,  $\lambda - 1$  is a factor of  $P(\mathcal{H}, \lambda)$ .  $\square$

Recall that for a hypergraph  $\mathcal{H}$ ,  $\mathcal{H} + K_1$  is the hypergraph obtained from  $\mathcal{H}$  by adding a new vertex  $u$  and adding new edges  $\{u, v\}$  for all vertices  $v$  in  $\mathcal{H}$ . By Proposition 8, the following result is obtained.

**Corollary 4.** *For any hypergraph  $\mathcal{H}$ ,  $P(\mathcal{H} + K_1, \lambda) = \lambda P(\mathcal{H}, \lambda - 1)$ .*

By Corollary 4,  $P(\mathcal{H} + K_1, \lambda)$  has a factor  $(\lambda - 1)^2$  if and only if  $P(\mathcal{H}, \lambda)$  has a factor  $\lambda^2$ . If  $\mathcal{H}$  is connected, then  $\mathcal{H} + K_1$  is not separable. By Theorem 3, there exist non-separable hypergraphs whose chromatic polynomials have a factor  $(\lambda - 1)^2$ .

Now we establish two important results for proving Theorem 4.

**Proposition 9.** *Let  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  be any connected hypergraph with a vertex  $w$  and two proper subsets  $\mathcal{V}_1$  and  $\mathcal{V}_2$  of  $\mathcal{V}$  such that  $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ ,  $\mathcal{V}_1 \cap \mathcal{V}_2 = \{w\}$  and for each  $e \in \mathcal{E}$ , either  $w \in e$  or  $e \subseteq \mathcal{V}_i$  for some  $i$ . Then  $(\lambda - 1)^2$  is a factor of  $P(\mathcal{H}, \lambda)$  if and only if  $\lambda^2$  is a factor of the following polynomial:*

$$\sum_{\mathcal{V}_0 \in \mathcal{I}_{(\mathcal{V}_1, \mathcal{V}_2)}(\mathcal{H})} P(\mathcal{H} - \mathcal{V}_0, \lambda), \quad (26)$$

where  $\mathcal{I}_{(\mathcal{V}_1, \mathcal{V}_2)}(\mathcal{H})$  is the set of those  $\mathcal{V}_0 \in \mathcal{I}(\mathcal{H})$  with  $\mathcal{V}_i \subseteq \mathcal{V}_0$  for some  $i \in \{1, 2\}$ .

*Proof.* By Proposition 8,

$$P(\mathcal{H}, \lambda) = \lambda \sum_{\mathcal{V}_0 \in \mathcal{I}_{(\mathcal{V}_1, \mathcal{V}_2)}(\mathcal{H})} P(\mathcal{H} - \mathcal{V}_0, \lambda - 1) + \lambda \sum_{\substack{w \in \mathcal{V}_0 \in \mathcal{I}(\mathcal{H}) \\ \mathcal{V}_0 \notin \mathcal{I}_{(\mathcal{V}_1, \mathcal{V}_2)}}} P(\mathcal{H} - \mathcal{V}_0, \lambda - 1). \quad (27)$$

For any  $\mathcal{V}_0 \in \mathcal{I}(\mathcal{H})$  with  $w \in \mathcal{V}_0$ , if  $\mathcal{V}_0 \notin \mathcal{I}_{(\mathcal{V}_1, \mathcal{V}_2)}(\mathcal{H})$ , then  $\mathcal{H} - \mathcal{V}_0$  is disconnected and Corollary 3 implies that  $\lambda^2$  is a factor of  $P(\mathcal{H} - \mathcal{V}_0, \lambda)$  and thus  $(\lambda - 1)^2$  is a factor of  $P(\mathcal{H} - \mathcal{V}_0, \lambda - 1)$ . Hence the result follows from (27).  $\square$

Let  $\mathcal{I}'_{(\mathcal{V}_1, \mathcal{V}_2)}(\mathcal{H})$  be the set of those members  $\mathcal{V}_0$  in  $\mathcal{I}_{(\mathcal{V}_1, \mathcal{V}_2)}(\mathcal{H})$  such that  $\mathcal{H} - \mathcal{V}_0$  is connected. For each  $\mathcal{V}_0 \in \mathcal{I}_{(\mathcal{V}_1, \mathcal{V}_2)}(\mathcal{H}) - \mathcal{I}'_{(\mathcal{V}_1, \mathcal{V}_2)}(\mathcal{H})$ ,  $\mathcal{H} - \mathcal{V}_0$  is disconnected and thus  $P(\mathcal{H} - \mathcal{V}_0, \lambda)$  has a factor  $\lambda^2$  by Corollary 3. By Proposition 9, we get the following result.

**Corollary 5.** *Let  $\mathcal{H} = (\mathcal{V}, \mathcal{E})$  be any connected hypergraph with a vertex  $w$  and two proper subsets  $\mathcal{V}_1$  and  $\mathcal{V}_2$  of  $\mathcal{V}$  such that  $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ ,  $\mathcal{V}_1 \cap \mathcal{V}_2 = \{w\}$  and for each  $e \in \mathcal{E}$ , either  $w \in e$  or  $e \subseteq \mathcal{V}_i$  for some  $i$ . Then  $(\lambda - 1)^2$  is a factor of  $P(\mathcal{H}, \lambda)$  if and only if  $\lambda^2$  is a factor of*

$$\sum_{\mathcal{V}_0 \in \mathcal{I}'_{(\mathcal{V}_1, \mathcal{V}_2)}(\mathcal{H})} P(\mathcal{H} - \mathcal{V}_0, \lambda). \quad (28)$$

We are now going to prove Theorem 4.

*Proof of Theorem 4:* (i) First assume that  $F(\mathcal{H}) = \{w\}$ . As  $|\mathcal{E}| \geq 2$  and  $\mathcal{H}$  is Sperner,  $|\mathcal{V}| \geq 3$ . Then  $\mathcal{H}$  is separable at  $w$ . Assume that  $\mathcal{V}_1$  and  $\mathcal{V}_2$  are proper subsets of  $\mathcal{V}$  such that  $\mathcal{V}_1 \cap \mathcal{V}_2 = \{w\}$  and  $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ . As  $w$  is the only member in  $F(\mathcal{H})$ ,  $\mathcal{V} - \{w\} \notin \mathcal{I}(\mathcal{H})$  for every  $u \in \mathcal{V} - \{w\}$ . Thus, for each  $\mathcal{V}_0 \in \mathcal{I}(\mathcal{H})$  with  $w \in \mathcal{V}_0$ , we have  $|\mathcal{V}_0| \leq |\mathcal{V}| - 2$  and so  $\mathcal{H} - \mathcal{V}_0$  is an empty graph of order at least 2, implying that  $\lambda^2$  is a factor of  $P(\mathcal{H} - \mathcal{V}_0, \lambda)$ . By Proposition 8 or Proposition 9,  $(\lambda - 1)^2$  is a factor of  $P(\mathcal{H}, \lambda)$ .

Now consider the case that  $k = |F(\mathcal{H})| \geq 2$ . By (5), we have

$$P(\mathcal{H}, \lambda) = P(\mathcal{H} + F(\mathcal{H}), \lambda) + P(\mathcal{H} \cdot F(\mathcal{H}), \lambda). \quad (29)$$

As  $F(\mathcal{H}) \subseteq e$  for each  $e \in \mathcal{E}$ , Proposition 1 implies that

$$P(\mathcal{H} + F(\mathcal{H}), \lambda) = P(\mathcal{H}_0, \lambda) = \lambda^{|\mathcal{V}| - k}(\lambda^k - \lambda), \quad (30)$$

where  $\mathcal{H}_0$  is the hypergraph with vertex set  $\mathcal{V}$  and edge set  $\{F(\mathcal{H})\}$ . By (29) and (30),

$$P(\mathcal{H}, \lambda) = \lambda^{|\mathcal{V}| - k}(\lambda^k - \lambda) + P(\mathcal{H} \cdot F(\mathcal{H}), \lambda). \quad (31)$$

Observe that  $\mathcal{H} \cdot F(\mathcal{H})$  is Sperner, connected and has as many edges as  $\mathcal{H}$ . As  $\mathcal{H} \cdot F(\mathcal{H})$  has at least two edges and  $|F(\mathcal{H} \cdot F(\mathcal{H}))| = 1$ ,  $P(\mathcal{H} \cdot F(\mathcal{H}), \lambda)$  has a factor  $(\lambda - 1)^2$  by the result proved above. Since  $\lambda^{|\mathcal{V}| - k}(\lambda^k - \lambda)$  does not have a factor  $(\lambda - 1)^2$ , (31) implies that  $P(\mathcal{H}, \lambda)$  does not have a factor  $(\lambda - 1)^2$ .

(ii) As  $F(\mathcal{H}) = \emptyset$ , it is impossible that  $\mathcal{V}_i \in \mathcal{I}(\mathcal{H})$  for both  $i = 1, 2$ . By Proposition 9, if  $\mathcal{V}_i \notin \mathcal{I}(\mathcal{H})$  for both  $i = 1, 2$ , then  $(\lambda - 1)^2$  is a factor of  $P(\mathcal{H}, \lambda)$ .

Now we assume that  $\mathcal{V}_1 \in \mathcal{I}(\mathcal{H})$  but  $\mathcal{V}_2 \notin \mathcal{I}(\mathcal{H})$ . By Proposition 9,  $(\lambda - 1)^2$  is a factor of  $P(\mathcal{H}, \lambda)$  if and only if  $\lambda^2$  is a factor of the following polynomial:

$$\sum_{\mathcal{V}_0 \in \mathcal{I}_{\mathcal{V}_1}(\mathcal{H})} P(\mathcal{H} - \mathcal{V}_0, \lambda), \quad (32)$$

where  $\mathcal{I}_{\mathcal{V}_1}(\mathcal{H})$  is the set of those  $\mathcal{V}_0 \in \mathcal{I}(\mathcal{H})$  with  $\mathcal{V}_1 \subseteq \mathcal{V}_0$ . Observe that

$$\sum_{\mathcal{V}_0 \in \mathcal{I}_{\mathcal{V}_1}(\mathcal{H})} P(\mathcal{H} - \mathcal{V}_0, \lambda) = \sum_{\substack{\mathcal{V}_1 \cup \mathcal{V}' \in \mathcal{I}(\mathcal{H}) \\ \mathcal{V}' \subseteq \mathcal{V}_2 - \{w\}}} P(\mathcal{H} - (\mathcal{V}_1 \cup \mathcal{V}'), \lambda). \quad (33)$$

Let  $\mathcal{H}_0$  denote the hypergraph  $\mathcal{H} \cdot \mathcal{V}_1$  and let  $w_0$  denote the vertex in  $\mathcal{H}_0$  which is produced after identifying all vertices  $\mathcal{V}_1$  as one. Thus the vertex set of  $\mathcal{H}_0$  is  $(\mathcal{V}_2 - \{w\}) \cup \{w_0\}$ . Observe that for any  $\mathcal{V}' \subseteq \mathcal{V}_2 - \{w\}$ ,  $\mathcal{V}_1 \cup \mathcal{V}' \in \mathcal{I}(\mathcal{H})$  if and only if  $\{w_0\} \cup \mathcal{V}' \in \mathcal{I}(\mathcal{H}_0)$ , and  $\mathcal{H} - (\mathcal{V}_1 \cup \mathcal{V}')$  is exactly the hypergraph  $\mathcal{H}_0 - (\{w_0\} \cup \mathcal{V}')$ . Thus, by (33),

$$\sum_{\mathcal{V}_0 \in \mathcal{I}_{\mathcal{V}_1}(\mathcal{H})} P(\mathcal{H} - \mathcal{V}_0, \lambda) = \sum_{\mathcal{V}' \cup \{w_0\} \in \mathcal{I}(\mathcal{H}_0)} P(\mathcal{H}_0 - (\mathcal{V}' \cup \{w_0\}), \lambda). \quad (34)$$

By Proposition 8, the right-hand side of (34) has a factor of  $\lambda^2$  if and only if  $P(\mathcal{H}_0, \lambda)$  has a factor  $(\lambda - 1)^2$ .

Hence (ii) holds.  $\square$

For any graph  $G$ , if  $G$  has two edges  $e_1, e_2$  which have no any common end, then  $\mathcal{H}_{\bullet G}$  is separable at vertex  $w$  which is the vertex not in  $G$  and  $\mathcal{I}_{(\mathcal{V}_1, \mathcal{V}_2)}(\mathcal{H}_{\bullet G}) = \emptyset$  for suitable  $\mathcal{V}_1, \mathcal{V}_2$  with  $e_i \subseteq \mathcal{V}_i$  for  $i = 1, 2$ , implying that  $P(\mathcal{H}_{\bullet G}, \lambda)$  has a factor  $(\lambda - 1)^2$  by Proposition 9.

By Theorem 4, we can easily get examples of separable hypergraphs  $\mathcal{H}$  whose chromatic polynomials don't have a factor  $(\lambda - 1)^2$ . Let  $\mathcal{H}$  be a hypergraph with vertex set  $\{w\} \cup \{x_i : 1 \leq i \leq s\} \cup \{y_j : 1 \leq j \leq t\}$  and edge set

$$\{\{y_j : 1 \leq j \leq t\}\} \cup \{\{w, x_1, x_2, \dots, x_s, y_j\} : 1 \leq j \leq t\}, \quad (35)$$

where  $s \geq 1$  and  $t \geq 1$ . Then  $\mathcal{V}_1 = \{w\} \cup \{x_i : 1 \leq i \leq s\}$  is a member of  $\mathcal{I}(\mathcal{H})$  while  $\mathcal{V}_2 = \{w\} \cup \{y_j : 1 \leq j \leq t\}$  is not. By Theorem 4,  $P(\mathcal{H}, \lambda)$  has a factor  $(\lambda - 1)^2$  if and only if  $P(\mathcal{H} \cdot \mathcal{V}_1, \lambda)$  has a factor  $(\lambda - 1)^2$ . Observe that  $\mathcal{H} \cdot \mathcal{V}_1$  is a hypergraph with vertex set  $\{w\} \cup \{y_j : 1 \leq j \leq t\}$  and edge set

$$\{\{y_j : 1 \leq j \leq t\}\} \cup \{\{w, y_j\} : 1 \leq j \leq t\}. \quad (36)$$

By Corollary 4,

$$P(\mathcal{H} \cdot \mathcal{V}_1, \lambda) = \lambda P(\mathcal{H} - \mathcal{V}_1, \lambda - 1) = \lambda((\lambda - 1)^t - (\lambda - 1)), \quad (37)$$

which does not have a factor  $(\lambda - 1)^2$ .

Theorem 4 actually provides a method of producing a separable hypergraph  $\mathcal{H}$  from a given hypergraph  $\mathcal{H}'$  such that  $P(\mathcal{H}, \lambda)$  has a factor  $(\lambda - 1)^2$  if and only if  $P(\mathcal{H}', \lambda)$  has a factor  $(\lambda - 1)^2$ . For any connected and Sperner hypergraph  $\mathcal{H}'$  with at least

two edges and any vertex  $w$  in  $\mathcal{H}'$ , let  $\mathcal{H}$  be a connected hypergraph with vertex set  $\mathcal{V}(\mathcal{H}') \cup S$ , where  $S$  is a non-empty set, and edge set

$$(\mathcal{E}(\mathcal{H}') - \mathcal{E}') \cup \{e \cup S_e : e \in \mathcal{E}', S_e \subseteq S\}, \quad (38)$$

where  $\mathcal{E}'$  is a set of some edges  $e \in \mathcal{E}(\mathcal{H}')$  with  $w \in e$  and  $S_e$  is any subset of  $S$ . As  $\mathcal{H}$  must be connected,  $\bigcup_{e \in \mathcal{E}'} S_e = S$ . By Theorem 4,  $P(\mathcal{H}, \lambda)$  has a factor  $(\lambda - 1)^2$  if and only if  $P(\mathcal{H}', \lambda)$  has a factor  $(\lambda - 1)^2$ .

## 6 Further study

It is well known that for any simple graph  $G$ , the multiplicity of root “0” of  $P(G, \lambda)$  is equal to the number of components of  $G$ . However Theorem 3 shows that this property does not hold for hypergraphs. Regarding the multiplicity of root “0” of  $P(\mathcal{H}, \lambda)$  for a hypergraph  $\mathcal{H}$ , the following problems are worth for study.

**Problem 1.** *Let  $\mathcal{H}$  be a connected hypergraph.*

- (a) *What is a necessary and sufficient condition for  $P(\mathcal{H}, \lambda)$  to have a factor  $\lambda^2$ ?*
- (b) *Is it possible that the multiplicity of root “0” of  $P(\mathcal{H}, \lambda)$  is larger than 2 for some non-separable hypergraph  $\mathcal{H}$ ?*
- (c) *What is the relation between the multiplicity of root “0” of  $P(\mathcal{H}, \lambda)$  and the structure of  $\mathcal{H}$ ?*

Note that for a bridge  $e$  in a connected hypergraph  $\mathcal{H}$ , by Theorem 5,  $P(\mathcal{H}, \lambda)$  has a factor  $\lambda^2$  if and only if  $P(\mathcal{H}/e, \lambda)$  has a factor  $\lambda^2$ . Thus the study of Problem 1(a) can be focused on those connected hypergraphs without bridges.

It is also well known that for a connected graph  $G$ ,  $(\lambda - 1)^2$  is a factor of  $P(G, \lambda)$  if and only if  $G$  is separable (i.e.,  $G$  has a cut-vertex) (see [8]), and the multiplicity of root “1” of  $P(G, \lambda)$  is equal to the number of blocks of  $G$ . However, such result does not hold for hypergraphs. Theorem 4 shows that  $(\lambda - 1)^2$  is a factor of  $P(\mathcal{H}, \lambda)$  for some but not all connected and separable hypergraphs  $\mathcal{H}$ . For connected but non-separable hypergraphs, Theorem 3 and Corollary 4 imply their chromatic polynomials may also have a factor  $(\lambda - 1)^2$ .

Regarding the multiplicity of root “1” of  $P(\mathcal{H}, \lambda)$  for a connected hypergraph  $\mathcal{H}$ , we propose the following problem.

**Problem 2.** *Let  $\mathcal{H}$  be a connected hypergraph.*

- (a) *What is a necessary and sufficient condition for  $P(\mathcal{H}, \lambda)$  to have a factor  $(\lambda - 1)^2$ ?*

- (b) What is the relation between the multiplicity of root “1” of  $P(\mathcal{H}, \lambda)$  and the structure of  $\mathcal{H}$ ?

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